VOLATILITY ESTIMATION FOR STOCHASTIC PROJECT VALUE MODELS

Luiz E. Brandão
IAG Business School
Pontifícia Universidade Católica do Rio de Janeiro

James S. Dyer
McCombs School of Business
The University of Texas at Austin

Warren J. Hahn
Graziadio School of Business and Management
Pepperdine University

Abstract

Consolidation of multiple sources of uncertainty into a single stochastic process for project value can provide increased computational flexibility for the analysis of complex real option valuation problems. Nonetheless, the volatility, a critical parameter in the stochastic project value model, is systematically overstated by the existing estimation methods, which can result in incorrect option values. Several recently published works have discussed examples that illustrate this issue numerically, and have developed the rationale for addressing the root problem and provided revised estimation methods. In this article, we extend that work by first analytically proving both the source of the bias and the adjustment to remove it, and then by generalizing for the cases of levered cash flows and non-constant volatility. In each case, we show how a revised estimation methodology can be applied with example problems.

1 Corresponding author: Warren J. Hahn, Pepperdine University, 24255 Pacific Coast Highway, Malibu, California 90263; Tel. +1 310 506 8542; Fax: +1 310 506 4126; E-mail address: Joe.Hahn@pepperdine.edu
1 Introduction

Real options valuation methods have gained widespread acceptance in recent years as a complement to traditional asset pricing methods. The main reasons for this are the limitations of the discounted cash flow (DCF) methods in valuing projects that have significant managerial flexibility. Although methods based on option valuation principles may have great intuitive appeal, in practical terms, their use can sometimes be limited due to the mathematical complexity of these techniques and the difficulties of modeling more complicated real life problems. In response to these issues, Copeland and Antikarov (2001) proposed the use of discrete models based on a single variable; project value. This approach allows detailed modeling of project decision making under uncertainty without the need to create complex analytical models, greatly simplifying the numerical solution of real option valuation problems. A critical part of this approach, however, is the estimation of the volatility for the stochastic project value model, since this parameter effectively specifies the degree of uncertainty confronting the decision-maker.

In the case of financial options, the determination of the volatility for the stochastic process model of the underlying asset is relatively straightforward, since volatility can be estimated from market or historical data. However, for non-market traded assets such as real projects this data is usually unavailable, and there can be some question as to what the underlying asset should be in the first place. Copeland and Antikarov (2001) present an approach based on the notion that the present value of the project without options is the best unbiased estimator of the “market” value of the project, an assumption they term the
Marketed Asset Disclaimer (MAD). Thus, under this assumption, the project without options serves as the underlying asset in the option valuation model.

If the changes in the value of the project without options are then assumed to vary over time according to a Geometric Brownian Motion (GBM) stochastic process, which is a commonly used model for asset values, then the project options can be valued with traditional option pricing methods, such as the binomial lattice approach of Cox, Ross and Rubinstein (1979). A GBM process for a stochastic variable $S$ has the form:

$$dS = \alpha S dt + \sigma S dz,$$

where $\alpha$ is a drift rate (mean return or growth rate), $\sigma$ is the process volatility (standard deviation of returns), and $dz = \epsilon \sqrt{dt}$ where $\epsilon$ follows a standard Wiener process. To specify a GBM model for a valuation model, we must therefore first estimate the drift rate and volatility.

The Copeland and Antikarov (2001) approach utilizes Monte Carlo simulation to estimate the volatility of the GBM process when the underlying asset is the project value without options. The procedure begins with the project pro forma worksheet using expected values for project uncertainties, which is used to calculate the discounted net present value $V_0$ of the project in time period 0. Next, key project uncertainties are entered as simulation input variables in the project cash flow pro-forma spreadsheet, so that each iteration in a simulation of the worksheet provides a new randomly sampled set of future cash flows from which the project value at the end of the first period $\tilde{V}_1$ is computed. Then a sample of the random variable $\tilde{V}$ can be calculated as
\[
\hat{\gamma} = \ln\left(\frac{\bar{V}_1}{V_0}\right)
\]  
(2)

where \(\hat{\gamma}\) is the project return between time zero and time 1. The estimate of the standard deviation of \(\hat{\gamma}\), denoted as \(s\), can be obtained from the simulation results. The project volatility \(\sigma\) is then defined as the annualized percentage standard deviation of the returns, and is estimated from the relationship \(\frac{s}{\sqrt{\Delta t}}\) where \(\Delta t\) is the length of the time period in years used in the cash flow pro forma worksheet. If the time period between \(\bar{V}_1\) and \(V_0\) is one year, then \(\sigma = s\).

This approach can be illustrated with the simple five-period example shown in Figure 1. We assume in this example that the project is subject to a single source of revenue uncertainty that is approximated by a GBM stochastic process with a growth rate \(\alpha = 6\%\) and a volatility \(\sigma_S = 25\%\), variable costs equal to 30\% of revenues, and a fixed cost of $3,000 in each of the five years of the project. We also assume that the firm will use a risk adjusted discount rate of \(\mu = 10\\%\). Given the initial estimate of the revenue in Period 0 of $10,000, the GBM process is approximated in discrete time in Excel by setting the revenue estimate in Period \(t+1\) equal to the revenue estimate in Period \(t\) times

\[
\exp\left(v\Delta t + \sigma_S NORMINV(RAND(0,1),0,\sqrt{\Delta t})\right) \quad \text{where} \quad v = \alpha - \sigma_S^2/2 \quad \text{and} \quad \Delta t = 1 \quad \text{in this case}
\]

since the time periods are one year.\(^2\) While this example project is very simple, it is representative of pro forma spreadsheet models used in practice. Further, Monte Carlo

\(^2\) Note that the Excel function used to approximate the GBM process in each period can be simplified if a commercial Monte Carlo simulation add-in such as @Risk or Crystal Ball is used. For example, the Excel function using @Risk would simply be \(\exp(RISKNORMAL(v\Delta t, \sigma_S\sqrt{\Delta t}))\).
simulation is commonly used with these types of spreadsheets as suggested above for the purpose of risk analysis (e.g., see Titman and Martin (2008)).

<table>
<thead>
<tr>
<th>$ 1000</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10,000</td>
<td>10,600</td>
<td>11,236</td>
<td>11,910</td>
<td>12,625</td>
<td>13,382</td>
<td></td>
</tr>
<tr>
<td>Variable Costs 30%</td>
<td>(3,180)</td>
<td>(3,371)</td>
<td>(3,573)</td>
<td>(3,787)</td>
<td>(4,015)</td>
<td></td>
</tr>
<tr>
<td>Fixed Costs</td>
<td>(3,000)</td>
<td>(3,000)</td>
<td>(3,000)</td>
<td>(3,000)</td>
<td>(3,000)</td>
<td></td>
</tr>
<tr>
<td>Investment</td>
<td>(20,000)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Free Cash Flow</td>
<td>(20,000)</td>
<td>4,420</td>
<td>4,865</td>
<td>5,337</td>
<td>5,837</td>
<td>6,368</td>
</tr>
</tbody>
</table>

\[ V_0 = 19,990 \]
\[ V_1 = 21,989 \]
\[ \text{Invest.} = (20,000) \]
\[ \text{NPV}_0 = (10) \]
\[ \mu = 10\% \]
\[ \sigma = 25\% \]

**Revenue Parameters**
- Growth \( \alpha = 6.0\% \)
- Volatility \( \sigma = 25\% \)

**Figure 1 – Example Project with Variable and Fixed Costs**

In Figure 1, \( V_0 = 19,990 \) is the present value of the expected cash flows from Period 1 to Period 5, discounted one year to Period 0, while \( V_1 = 21,989 \) is the present value of the expected cash flows discounted to Period 1. The volatility of this project can be estimated by Monte Carlo simulation with this pro forma sheet and using Eq. (2), where \( V_0 \) is a constant with the value shown above, but \( V_1 \) is a random variable which is recalculated in each iteration of the simulation, based on the realizations of the stochastic cash flows in Periods 1 through 5.

Given the estimate of the volatility associated with the GBM process, a binomial lattice or tree can be used to approximate the evolution of the project value over time in a manner that is consistent with the project cash flows using, for example, the approach of Cox et al. (1979). This type of lattice can then be used to value real options associated with
the project as discussed by Copeland and Antikarov and by Brandao, Dyer and Hahn (2005a).

Several researchers have subsequently utilized this approach to volatility estimation in their work. Herath and Park (2002) apply this approach to the case of compound options with several uncorrelated underlying variables and illustrate this with a multi-stage R&D investment problem. Cobb and Charnes (2004) extend this concept further by investigating the case of correlated inputs. Unfortunately, however, there is an issue with this approach; as noted by Smith (2005), the volatility estimates are upwardly biased, which causes the project value model to overstate the variance of project cash flows.

Brandao, Dyer and Hahn (2005b) used an alternative approach to volatility estimation which isolates the uncertainty resolved within each period in the simulation model and eliminates this bias. This work also demonstrated how the volatility, rather than being constant throughout the life of the project, may change from period to period. The main focus of that paper, however, was on the implementation of the corrected stochastic project value model in a binomial decision tree format, which can accommodate the non-constant volatility, rather than on the volatility estimation procedure.

Godinho (2006) reaches the same conclusion regarding the bias in the Copeland and Antikarov volatility estimation procedure, and develops a similar modification to remove the bias, as well as an enhanced technique which uses regression to reduce the complexity of the volatility estimation process. However, only the first period volatility estimate is evaluated in the primary example in that work, and therefore the case of non-constant volatility is not explored in detail. Furthermore, although Brandão, Dyer and Hahn (2005b) and Godinho (2006) show examples of the volatility estimation bias under the Copeland
and Antikarov method and provide the rationale for removing the bias, neither contain rigorous analytical proofs of the bias or the method for addressing it.

In this paper, we first show the source of the upward bias in the Copeland and Antikarov volatility estimation methodology, beginning with the stochastic project value model formulation and deriving the variance of the resultant cash flows. We then explicitly prove the existence of an unbiased estimation approach and demonstrate its application both analytically and empirically to a range of problem types.

This paper is organized as follows. In section 2 we present an overview of the problem of real option valuation and the determination of the project volatility for a simple project. In section 3 we evaluate the approach to estimating the project volatility proposed by Copeland and Antikarov (2001), comparing the results from that approach to the analytically derived volatility for a simple project. We then discuss an unbiased estimate of project volatility in section 4, and extend this approach to the case with leveraged cash flows in section 5. Section 6 illustrates the application of this approach to a practical example, and in section 7 we conclude with a brief discussion.

2 Volatility of Project Value

We first consider a simple project that is subject to a single source of uncertainty \( S(t) \), such as the revenue from selling a product with stochastic price, and only variable costs, \( C(t) = cS(t) \), where \( c < 1 \) is a constant. We assume that this project uncertainty follows a GBM diffusion process, as shown in Eq. (1). This simplification of only a single source of uncertainty allows us to obtain analytic results that provide insights regarding the issue of
We let $F(t)$ be the project cash flows, so that $F(t) = S(t) - cS(t) = (1-c)S(t)$, and define $k = (1-c)$. Since $F(t)$ is a linear function of $S(t)$, we can then determine the process for $F$ by a simple application of Itô’s Lemma, obtaining

$$dF = \alpha F \, dt + \sigma_S \, F \, dz$$  \hspace{1cm} (3)$$

where $F = kS$.

If $F(t)$ is a linear function of the underlying uncertainty $S(t)$, then the diffusion process of $F(t)$ is also a GBM with the same parameters as $S(t)$. This implies that for the simple project we have described, the volatility of the cash flows is equal to the volatility of the underlying uncertainty $S(t)$, and is independent of $k$.

Since the present value of the project at time $t$, $V_t$, is a function of the project cash flows, we next consider the stochastic model for project value. At time $t = 0$, the present value of the cash flow stream $F$ using the discount rate $\mu$ is $V_0 = \int_{t=0}^{\infty} E[F(t)] e^{-\mu t} \, dt$. Since $E[F(t)] = F_0 e^{\alpha t}$, then $V_0 = \int_{t=0}^{\infty} (F_0 e^{\alpha t}) e^{-\mu t} \, dt = \frac{F_0}{\mu - \alpha}$, $\mu > \alpha$. In general, with a drift rate of $\alpha$ and discount rate of $\mu$, the relationship between $V_t$ and $F_t$ is $V_t = \frac{F_t}{\mu - \alpha}$. Given that the cash flows $F$ of this simple project follow the stochastic process $dF = \alpha F \, dt + \sigma_S \, F \, dz$, we can apply Itô’s Lemma to obtain the diffusion process for $V$,

$$dV = \left( \frac{\partial V}{\partial F} \alpha F + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial F^2} \sigma_S^2 F^2 \right) \, dt + \frac{\partial V}{\partial F} \sigma_S F \, dz$$

which simplifies to
Therefore, we write the stochastic process for project value $V$ as:

$$dV = \alpha V dt + \sigma_S V dz$$  \hspace{1cm} (4)$$

Eq. (4) shows that the project value volatility is the same as the volatility of the cash flows, which is also the same as the volatility of the underlying uncertainty in this case. Note that the project value volatility does not depend on $k$, the fraction that relates the variable costs to the revenues. These results give us a basis for evaluating volatility estimation methods in the following two sections.

3 Volatility Estimation with Simulation

To evaluate the simulation approach for estimating volatility introduced by Copeland and Antikarov (2001), let us first assume that at time $t = 0$ we want to determine the volatility of a single period project, as depicted in Figure 2, with revenues $S(t)$ that follow a GBM as in Eq. (1). We also maintain the assumption that expected cash flows $F(t)$ are a constant proportion of the revenues so that $F(t) = kS(t)$ and that there are no fixed costs. The risk adjusted project discount rate is $\mu$. As we have seen previously, both the stochastic process for $F(t)$ and that of the project value $V(t)$ follow a GBM.

```
0  1
S_0  S_1
F_0  F_1
V_0  V_1
```

Figure 2 – Single Period Project
The volatility of an asset is defined as the annualized standard deviation of the returns, which we define as \( \ln(S) \). Thus, if \( G = \ln(S) \) are the returns of an asset \( S \) that follows a GBM, then by Itô’s Lemma the stochastic process for \( G \) is 
\[
dG = \left( \alpha - \frac{1}{2} \sigma_S^2 \right) dt + \sigma_S dz.
\]
This stochastic process is an arithmetic Brownian motion (ABM) with a volatility of \( \sigma_S \), which is the same as the volatility of \( S \). The expected return for the asset \( S \) is \( E[\tilde{\alpha}] = \alpha \), and we denote the returns in stochastic form as \( \tilde{\alpha} = \tilde{G} = \ln \tilde{S} \).

Copeland and Antikarov (2001) define the project volatility as the standard deviation of the project returns \( \gamma \), where 
\[
\tilde{\gamma} = \ln \left( \frac{\tilde{V}_1}{V_0} \right), \quad V_0 \text{ is the expected present value of the project at time 0 and } (1 ) \sum_n \left( V_F \right) e^{-\mu t} = \text{the stochastic project value at time 1 written in discrete time. The standard deviation is then determined by using Monte Carlo simulation.}
\]
Since we have seen previously that the project volatility must be the same as the volatility of the cash flows, it must also be the case that \( \text{Var}(\tilde{\gamma}) = \text{Var}(\tilde{\alpha}) \). To check this, we will determine the expression for \( \tilde{\gamma} \) and derive the variance and standard deviation analytically.

Using the GBM process described in Eq. (1), the process for the cash flow is defined by Eq. (3). The stochastic cash flow in \( t = 1 \) is then \( \tilde{F}_1 = F_0 e^{\tilde{\alpha}} \), and the expected value of this cash flow at time 0 is \( E_0 [\tilde{F}_1] = F_0 e^{\tilde{\alpha}} \). Accordingly, the stochastic project value \( \tilde{V}_1 \) for this one period project is \( \tilde{V}_1 = \tilde{F}_1 = F_0 e^{\tilde{\alpha}} \), and the expected value of the project is 
\[
V_0 = E_0 [\tilde{F}_1] e^{-\mu} = F_0 e^{\tilde{\alpha} - \mu}. \quad \text{The project returns can therefore be written as}
\]
\[ \tilde{y} = \ln \left( \frac{V_1}{V_0} \right) = \ln \left( e^{\alpha - \alpha + \mu} \right) \] 

The right hand side of this equation is simply \( \tilde{\alpha} - \alpha + \mu \), and since the constants do not affect the variance of the uncertain variable, \( \text{Var}(\tilde{y}) = \text{Var}(\tilde{\alpha}) \), which shows that the Copeland and Antikarov simulation approach is correct for a single period project.

However, let us now assume that the project has two periods, as shown in Figure 3.

![Figure 3 – Two Period Project](image-url)

In this project, the deterministic and stochastic expressions for cash flows for period 1 are \( F_1 = F_0 e^\alpha \) and \( \tilde{F}_1 = F_0 e^{\tilde{\alpha}_0} \), respectively. Similarly, for period 2, we have \( F_2 = F_1 e^\alpha = F_0 e^{\tilde{\alpha} + \alpha} \) and \( \tilde{F}_2 = F_1 e^{\tilde{\alpha}_1} = F_0 e^{\tilde{\alpha}_0 + \tilde{\alpha}_1} \) where \( \tilde{\alpha}_0 \) and \( \tilde{\alpha}_1 \) are the i.i.d. stochastic returns for the asset for the first (0 to 1) and second (1 to 2) periods, respectively. Copeland and Antikarov model the stochastic cash flow \( \tilde{F}_2 \) as a function of the uncertainty in both periods 1 and 2. Using this approach, the expected value of the project in \( t = 0 \) will be

\[ V_0 = F_0 e^{\alpha - \mu} \left( 1 + e^{\alpha - \mu} \right) \]  

(5)

The stochastic value of the project in period 1 can be expressed as the sum of the stochastic cash flows in period 1 and the discounted stochastic cash flows of period 2, \( \tilde{V}_1 = \tilde{F}_1 + \tilde{F}_2 e^{-\mu} \), all of which we can again write in terms of the period zero cash flows as:

\[ \tilde{V}_1 = F_0 \left( e^{\tilde{\alpha}_0} + e^{\tilde{\alpha}_0 + \tilde{\alpha}_1 - \mu} \right) \]  

(6)
We can then use Eq. (5) and Eq. (6) to write the uncertain rate of return $\tilde{\gamma} = \ln \left( \frac{\hat{V}_t}{V_0} \right)$ as

$$\tilde{\gamma} = \ln \left[ \frac{\mathcal{F}_0 \left( e^{\tilde{\alpha}_0} + e^{\tilde{\alpha}_1 + \tilde{\mu}} \right)}{\mathcal{F}_0 e^{\tilde{\mu}} (1 + e^{\tilde{\alpha} - \mu})} \right].$$

The variance of the returns $\tilde{\gamma}$ can then be expressed as

$$Var(\tilde{\gamma}) = Var \left( \tilde{\alpha}_0 + \ln \left[ 1 + e^{\tilde{\alpha}_1 - \mu} \right] \right) \quad (7)$$

The result in Eq. (7) shows that we obtain an upwardly biased estimate of the true variance for the project value.

To summarize, when the volatility of the project value between periods 0 and 1 is estimated using the uncertainty in period 1 for a single period project, we will obtain the correct answer using simulation. However, when this volatility is estimated for a two period project, using the discounted value of the uncertain returns in periods 1 and 2 overestimates the project volatility as $Var(\tilde{\gamma}) > Var(\tilde{\alpha}_0)$ since $\ln \left[ 1 + e^{\tilde{\alpha}_1 - \mu} \right] > 0$ are independent, which is consistent with the empirical observations made by Smith (2005). It is easy to see that this problem would be compounded for a project with three or more periods, so we should expect the error in the estimate of the volatility of a project to increase as a function of $t$ when this simulation approach is used.

Although this estimation procedure seems intuitively appealing, there is a logical inconsistency in including the uncertainty of the outcome in period 2. To understand why, we first recognize that the simulation process models possible realizations of future cash flows at a given time $t$, so that $\mathbb{E}_t \left[ \hat{V}_t \right]$ is taken with respect to the relevant downstream (future) uncertainty that may affect the value of $V_t$. Thus, in our example, each iteration of the simulation of project value at time $t = 1$ provides a new value $\mathbb{E}_1 \left[ \hat{V}_1 \right] = F_1 + \mathbb{E}_1 \left[ F_2 \right] e^{-\mu}$. 
However, the first step of the discrete process is only intended to model uncertainty in the first year of the project, and information concerning actual cash flow realized in period 2 is not yet available. Because of this the best unbiased estimate of this future cash flow is its expected value at time $t = 1$.

4 An Unbiased Estimate of Project Volatility

To demonstrate how to correct this bias, we use the same simple project defined in Section 3 and shown in Figure 2. For a single period, the results are exactly the same; that is, we trace the same steps shown in the previous section. However, we must develop a different approach when we extend to the two period project shown in Figure 3.

Since we again want to determine the annual volatility of the project we use the project values between periods 0 and 1; however we limit our simulation of uncertainty to this period only. The intuition for doing this can be seen by considering the analogy to the method of determining the volatility for a stock with price $P_t$. In that case, we calculate returns as $\ln \left( \frac{P_{t+1}}{P_t} \right)$ over a time series of historical prices, and estimate the volatility as the standard deviation of these returns, where the market price $P_t$ in any period $t$ is a function of the expected future returns for the stock based on current information available at time $t$. Therefore, in the above expression, the value $P_{t+1}$ does not include any information about the actual realized price in the next period, $t+2$, or any other future period.

Similarly, the period 1 uncertainty will generate stochastic values, $\tilde{F}_i = F_0 e^{\tilde{\alpha}}$, at the end of that period, but since no further uncertainty can yet be resolved, all subsequent cash
flows will be conditional expected values such that \( \hat{F}_2 = \hat{F}_1 e^\alpha \), and so on, since the best unbiased estimate of \( F_2 \) is the conditional expectation of \( F_2 \) given the realized value of \( F_1 \).

As before, we can write the value in period zero as shown in Eq. (5). However, by isolating the uncertainty in period 1, we now have a different expression for the value in period 1,
\[
\hat{V}_1 = \hat{F}_1 + \hat{F}_2 e^{-\mu} = F_0 e^{\alpha_0} + F_0 e^{\alpha_0 e^{-\mu}}
\]
which can be simplified to:
\[
\hat{V}_1 = F_0 e^{\alpha_0} \left(1 + e^{\alpha - \mu}\right) \tag{7}
\]

Eq. (7) represents the correct expression for \( \hat{V}_1 \) for the purpose of estimating the project volatility, in contrast to (6) which incorporated the additional uncertainty associated with the cash flow in period 2. Substituting (7) into the expression for project returns (2), we obtain
\[
\tilde{y} = \ln \left[ \frac{\hat{V}_1}{V_0} \right] = \tilde{\alpha}_0 - \alpha + \mu.
\]
Since the addition of a constant to a random variable does not change the variance, the variance of the returns is then \( \text{Var}(\tilde{y}) = \text{Var}(\tilde{\alpha}_0) \). This result indicates that in the BDH approach, as expected, the variance of the returns depends only on the variance of the drift rate \( \tilde{\alpha}_0 \) for this project.

5 Including a Leverage Effect

We now relax the assumptions for the simple project to include the operating leverage effect that arises when there are fixed costs associated with the project, in addition to the variable costs. This assumption is consistent with the structure of the example project with fixed and variable costs shown in Figure 1.

As before, we assume that the dynamics of the single source of project uncertainty \( S \) is governed by a GBM diffusion process in the form of Eq. (1). We also assume that
there is now a fixed cost $d$ such that the project cash flows $F$ are $F = kS - \omega$, $0 < k < 1$ and $\omega$ is a positive constant. Applying Itô's Lemma to Eq. (1) yields the process followed by the project cash flows, $dF = \alpha kS dt + \sigma_S kS dz$. Since $kS = F + \omega$, we can substitute to obtain $dF = \alpha(F + \omega) dt + \sigma_S (F + \omega) dz$, which shows that the volatility of the cash flows $F$ increases by a constant equivalent to $\omega \sigma_S$ due to the operating leverage when there are fixed costs. Again, we can see that the volatility term is independent of $k$, and that the effect of the fixed costs $\omega$ becomes less important as $F$ increases.

As before, the volatility of the project value $V$ can be determined from the process for the present value at any time $q = t$ of the expected cash flows $F_t$, which in this case is

$$V_t = \int_{q=t}^{\infty} E[F(q)] e^{-\mu q} dq = \int_{q=t}^{\infty} \left( (F_t + \omega) e^{\alpha q} - \omega \right) e^{-\mu q} dq,$$

or

$$V_t = \frac{F_t + \omega}{\mu - \alpha} - \frac{\omega}{\mu}$$

(8)

Applying Itô's Lemma to the process for project cash flows we obtain the following process for the project value $dV = \frac{(F + \omega)}{\mu - \alpha} \alpha dt + \frac{(F + \omega)}{\mu - \alpha} \sigma_S dz$. Using Eq. (8), we obtain

$$dV = \alpha \left( V + \frac{\omega}{\mu} \right) dt + \sigma_S \left( V + \frac{\omega}{\mu} \right) dz.$$  

If $X = \ln V$ is the process for the returns of the project, then Eq. (9) represents the dynamics of the project returns.

$$dX = \left[ \alpha \frac{V^*}{V} - \frac{1}{2} \sigma^2 \left( \frac{V^*}{V} \right)^2 \right] dt + \sigma_S \left( \frac{V^*}{V} \right) dz$$

(9)
where \( V^* = V + \frac{\omega}{\mu} \). This implies that in the presence of fixed costs, the volatility of the project value is extended by the term \( \frac{V^*}{V} \). While the volatility of the project value does not depend on the proportion \( k \) of the variable cost, it is a function of the fixed costs \( \omega \) and increases by an amount proportional to their discounted present value, \( \frac{\omega}{\mu} \). If this proportion changes throughout the full project life, the project volatility will not be constant. Thus, in general, the estimation process may involve the calculation of volatility in each period.

6 Application

We now apply the models developed in the previous sections to an example to demonstrate how this approach corrects the upward bias in the estimation of the project volatility, and also can be used to adjust for non-constant volatility. We assume in the first case in this example that there is a five period project, as shown in Figure 4, which is identical to the spreadsheet introduced in Figure 1 except that it does not have a fixed cost. Therefore, it is subject to a single source of revenue uncertainty approximated by a GBM stochastic process, with a growth rate \( \alpha = 6\% \) and a volatility \( \sigma_S = 25\% \), and variable costs \((1-k)\) equal to 30\% of revenues. The discount rate is \( \mu = 10\% \).
Since there are no fixed costs, the project volatility given by  
\[ dV = \alpha V dt + \sigma V dz \]
must be the same as the volatility of the revenues, or 25%, as we have seen in Section 4.

We ran a Monte Carlo simulation with 10,000 iterations using the Copeland and Antikarov approach with the discrete approximation to the GBM process for the revenue with uncertainty modeled in all periods, This provides an estimated value of \( \sigma_S = 36.1\% \), which overestimates the true project volatility, as we would expect given the results from Section 3.

To implement our approach, we make the revenue in Period 1 uncertain using the discrete approximation formula for a GBM with the parameters given above, but the formulae for the revenues in Periods 2 through 5 are simply the expected values of those cash flows, which in this case are equal to the outcomes in the previous periods escalated by the growth rate of 6%. A Monte Carlo simulation with 10,000 iterations using our approach provides the correct value of \( \sigma_S = 25\% \).

We next consider the case where the project also has fixed cost of $3,000 in each of the five years of the project, which is the example introduced in Figure 1. According to Eq.
Eqs. (9), in the presence of fixed costs the project volatility will be factored by \( \frac{V + \omega/\mu}{V} \), where the term \( \omega/\mu \) represents the perpetuity value of the fixed costs. Since the project has a life of only five years, the present value of the fixed costs during five years at 10% is $11,372. The project volatility predicted analytically is \( \sigma = (19,990 + 14,372)/19,990 \times 25\% = 42.97\% \), treating the five-year present value of the fixed costs as though it represents the value of a perpetuity. The fact that apply the equation derived for perpetual cash flows to the finite life case implies that there will be an approximation error. Nonetheless we will see that this error is quite small, as illustrated in Table 1.

The simulation results provide a value of 42.5% for our approach (BDH), which is the correct estimate of the volatility for this project. This result is within an error margin of approximately 1% of the corresponding analytic value. On the other hand, the Copeland and Antikarov (CA) approach provides a simulation result of 66.8%. Since Eq. (9) was derived assuming perpetual cash flow streams, increasing the number of periods decreases the discrepancy between the BDH simulation and the analytic results based on the perpetuity assumption as can be seen in Table 1.

<table>
<thead>
<tr>
<th>Number of Periods</th>
<th>Vol Analytic</th>
<th>Vol BDH</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>43.0%</td>
<td>42.5%</td>
</tr>
<tr>
<td>10</td>
<td>38.7%</td>
<td>39.0%</td>
</tr>
<tr>
<td>20</td>
<td>35.0%</td>
<td>35.2%</td>
</tr>
<tr>
<td>50</td>
<td>31.5%</td>
<td>31.5%</td>
</tr>
<tr>
<td>100</td>
<td>30.5%</td>
<td>30.5%</td>
</tr>
<tr>
<td>200</td>
<td>30.3%</td>
<td>30.3%</td>
</tr>
<tr>
<td>Infinite</td>
<td>30.3%</td>
<td>30.3%</td>
</tr>
</tbody>
</table>

Table 1 – Comparison of analytic and simulation results for BDH
The discussion in this example has been about the volatility estimate for the first period (Time 0 to Time 1). If indeed the assumption is valid that the changes in the value of the project follow a GBM distribution over time, then the volatility would be constant and the estimated volatility for the first period would apply to all subsequent periods. However, as discussed earlier, for some applications this assumption may not be appropriate, in which case the volatility would be specific to each time period. Copeland and Antikarov (2001, Chapter 12) make this same point but do not illustrate how to modify the Monte Carlo simulation procedure to obtain period-specific volatility estimates. We assume their approach would be to use the same output variable as in Eq. (2), but to apply it to subsequent periods, while continuing to model all values downstream of the reference period as stochastic.

To illustrate a case in which the volatility may change in each period, in the final version of this example we change the stochastic process for revenue from a GBM with constant variance to a mean-reverting process, and also add a stochastic jump to the variable cost process (Figure 5). The mean-reverting process for revenue is a simple one factor process (Ornstein-Uhlenbeck) of the form

\[ dR_t = \eta (\bar{R} - R_t) dt + \sigma_d dz_t, \]

where \( R_t \) is the log of revenue, \( \eta \) is the mean reversion coefficient, \( \bar{R} \) is the log of the long term equilibrium revenue, \( \sigma_d \) is the revenue process volatility and \( z \) follows a Weiner process. To model variable costs, we use a simple discrete jump process for the variable cost ratio, which has an initial value of 30% of revenue. We assume that this ratio will remain constant until a one-time technology breakthrough reduces it to 15%, and that there is a 50% probability of the breakthrough occurring in each year until it occurs. These modifications might apply to a real example where revenue depends on a price that reaches
(reverts to) some equilibrium level and variable costs are affected by changes in technology.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>5,000</td>
<td>6,568</td>
<td>7,749</td>
<td>8,567</td>
<td>9,105</td>
<td>9,447</td>
</tr>
<tr>
<td>Variable Costs</td>
<td>(1.478)</td>
<td>(1.453)</td>
<td>(1.445)</td>
<td>(1.451)</td>
<td>(1.461)</td>
<td></td>
</tr>
<tr>
<td>Fixed Costs</td>
<td>(1,500)</td>
<td>(1,500)</td>
<td>(1,500)</td>
<td>(1,500)</td>
<td>(1,500)</td>
<td></td>
</tr>
<tr>
<td>Investment</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td></td>
</tr>
<tr>
<td>Free Cash Flow</td>
<td>(10,000)</td>
<td>3,590</td>
<td>4,796</td>
<td>5,621</td>
<td>6,164</td>
<td>6,486</td>
</tr>
<tr>
<td>PV&lt;sub&gt;0&lt;/sub&gt; =</td>
<td>19,681</td>
<td>21,649</td>
<td>19,865</td>
<td>16,575</td>
<td>12,050</td>
<td>6,486</td>
</tr>
<tr>
<td>Invest. =</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td>(10,000)</td>
<td></td>
</tr>
<tr>
<td>NPV&lt;sub&gt;0&lt;/sub&gt; =</td>
<td>9,681</td>
<td>9,681</td>
<td>9,681</td>
<td>9,681</td>
<td>9,681</td>
<td></td>
</tr>
<tr>
<td>Reversion Speed ((\kappa)) =</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>(\sigma) =</td>
<td>30%</td>
<td>30%</td>
<td>30%</td>
<td>30%</td>
<td>30%</td>
<td></td>
</tr>
<tr>
<td>Long-term mean= $10,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Variable Cost Parameters**

- \(P(\text{Technology Change}) = 0.5\) (annually, until breakthrough)
- \(\text{Variable Cost Ratio} = 30\%\) (percentage of revenue)
- \(\text{Variable Cost Ratio} = 15\%\) (after technology breakthrough)

**Figure 5 – Example Project with Mean-Reverting Revenues and Modified Variable Costs**

We might expect the volatility to change from period to period in this example because of the two non-GBM processes that do not have variances that change in constant proportion with time. Fortunately, a multi-period analysis can be easily performed using our approach by defining the project volatility in *each* year \(i\) as the standard deviation of the returns \(\tilde{y}_i\), where \(\tilde{y}_i = \ln\left(\frac{V_{i+1}}{V_i}\right)\) and \(V_i\) is the project value in year \(i\). Thus, we estimate the volatility five times, once in each period, with the results shown in Table 2, along with the volatility estimated by the Copeland and Antikarov approach.
In an example such as this one, it becomes difficult, if not impossible, to calculate the volatility analytically for comparison and benchmarking. However, we can use the same simulation run in which we estimate the volatility for the binomial tree to numerically generate the distribution of cash flows in each period. Then, we build a binomial tree using the Cox et al. (1979) formulae with the different volatility estimates from Table 2 used in each period. This binomial tree also produces probability distributions for cash flows in each period (obtained by multiplying all of the endpoints by their respective cash flow values). By comparing the distributions, we can evaluate how well the binomial tree with the estimated period-by-period volatilities approximates the actual distribution of cash flows, as modeled by simulation.

We conducted this analysis for the example above, and focused on the cash flows in the last period. For all three methods – simulation and binomial trees using volatilities estimated using both the Copeland and Antikarov method and our approach, the mean value for Period 5 cash flow was $6,486. However the standard deviation of this cash flow, determined to be $2,466 by simulation, was $3,989 (62% higher) in the binomial tree with the volatility estimates from the Copeland and Antikarov approach. The standard deviation

<table>
<thead>
<tr>
<th>Period</th>
<th>Vol C&amp;A</th>
<th>Vol BDH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27.5%</td>
<td>15.6%</td>
</tr>
<tr>
<td>2</td>
<td>27.8%</td>
<td>18.0%</td>
</tr>
<tr>
<td>3</td>
<td>28.2%</td>
<td>20.9%</td>
</tr>
<tr>
<td>4</td>
<td>28.7%</td>
<td>24.7%</td>
</tr>
<tr>
<td>5</td>
<td>30.4%</td>
<td>30.4%</td>
</tr>
</tbody>
</table>

Table 2 – Yearly Volatility for Revised Example
of the Period 5 cash flow from the binomial tree using our approach was also slightly high, but was a much closer approximation to the simulation standard deviation, at $2,932. Additionally, we have plotted the three distributions in Figure 6 below, which shows the improved fit of the binomial tree with our estimates of period-by-period volatility, which had a root mean square error of 2.25%, compared to a 10.55% for the binomial tree with the Copeland and Antikarov volatility estimates.

![Figure 6 – Simulated vs. Binomial Tree Model Distributions for Year 5 Cash Flows](image)

This example illustrates the importance of using a volatility estimation approach that isolates the variability within each period and includes the flexibility to model changes in the magnitude of that variability across time. Even though the expected values of these distributions were identical using the three approaches that we compared, estimates of option value based on these binomial trees would be more accurate using volatility.
estimates derived using our approach, due to an improved approximation of the variance of the distributions of cash flows.

7 Conclusions

Volatility is a critical input variable for stochastic process models in real option valuation; however, as has been shown here and elsewhere, values derived via existing simulation-based estimation methods are biased high. In this paper we have analytically verified the source of this bias and confirmed that an approach that appropriately models the temporal resolution of uncertainty and updating of conditional expectations in each discrete time increment for problems is unbiased when the cash flows can be closely approximated by a GBM stochastic process.

Furthermore we have explicitly shown how this approach can be extended to cases with leveraged cash flows and non-constant volatility. The latter is an important consideration when the assumption that the project value varies according to a GBM stochastic process is not a reasonable one. This would be the case, for example, when there is a leveraging effect on a stochastic variable or when multiple stochastic variables are to be combined into a single underlying uncertainty in an option valuation model. Accordingly, by comparing the estimated values from simulation with our approach to the exact analytically derived values for an example with five time periods and two stochastic variables, we have verified that this approach provides accurate volatility estimates in each period. The use of this approach therefore resolves a critical error in the estimation of project volatility using Monte Carlo simulation methods, and ensures more accurate valuation in real option applications.
References:


